

Folded potentials in cluster physics - a comparison of Yukawa and Coulomb potentials with Riesz fractional integrals

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In cluster physics a single particle potential to determine the microscopic part of the total energy of a collective configuration is necessary to calculate the shell- and pairing effects. In this paper we investigate the properties of the Riesz fractional integrals and compare their properties with the standard Coulomb and Yukawa potentials commonly used. It is demonstrated, that Riesz potentials may serve as a promising extension of standard potentials and may be reckoned as a smooth transition from Coulomb to Yukawa like potentials, depending of the fractional parameter α . For the macroscopic part of the total energy the Riesz potentials treat the Coulomb-, symmetry- and pairing-contributions from a generalized point of view, since they turn out to be similar realizations of the same fractional integral at distinct α values.

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I. INTRODUCTION

Convolution integrals of the type

$$F(x) = \int_{-\infty}^{\infty} d\xi f(x - \xi) w(\xi) \quad (1)$$

$$= \int_{-\infty}^{\infty} d\xi f(\xi) w(x - \xi) \quad (2)$$

play a significant role in the areas of signal- and image processing or in the solution of differential equations.

In classical physics the first contact with the 3D-generalization of a convolution integral occurs within the framework of gravitation and electromagnetic theory respectively in terms of a volume integral to determine the potential V of a given charge density distribution ρ :

$$V(\vec{x}) = \int_{\mathbb{R}^3} d^3\xi \frac{\rho(\vec{\xi})}{|\vec{x} - \vec{\xi}|} \quad (3)$$

where the weight w

$$w(|\vec{x} - \vec{\xi}|) = \frac{1}{|\vec{x} - \vec{\xi}|} \quad (4)$$

is interpreted as the gravitational or electromagnetic field of a point charge [22, 24].

In nuclear physics collective phenomena like fission or cluster-radioactivity, where many nucleons are involved, are successfully described introducing the concept of a collective single-particle potential, based on folded potentials of e.g. Woods-Saxon type. Several weight functions have been investigated in the past.

In this paper we will demonstrate, that the fractional Riesz-potential [40, 41, 46] which extends the weight function introducing the fractional parameter α

$$w(|\vec{x} - \vec{\xi}|) = \frac{1}{|\vec{x} - \vec{\xi}|^\alpha} \quad (5)$$

serves as a serious alternative for commonly used Nilsson [38, 39], Woods-Saxon [8] and folded Yukawa potentials [1, 33], modeling the single particle potential widely applied in nuclear physics as well as in electronic cluster physics.

Hence we give a direct physical interpretation of a multi-dimensional fractional integral within the framework of fragmentation theory [31], which is the fundamental tool to describe the dynamic development of clusters in nuclear and atomic physics.

II. FOLDED POTENTIALS IN FRAGMENTATION THEORY

The use of collective models for a description of collective aspects of nuclear motion has proven considerably successful during the past decades.

Calculating life-times of heavy nuclei [15, 37, 53], fission yields [28], giving insight into phenomena like cluster-radioactivity [44], bimodal fission [18] or modeling the ground state properties of triaxial nuclei [35] - remarkable results have been achieved by introducing an appropriate set of collective coordinates, like length, deformation, neck or mass-asymmetry [30] for a given nuclear shape and investigating its dynamic properties.

As an example, in figure 1 the parametrization of the 3-spheres-model is sketched. It determines the geometry of a given cluster shape by 2 intersecting spheres, which

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are smoothly connected via a third sphere, which models a neck depending on the size of the radius r_3 .

The corresponding set of collective coordinates $\{q^i, i = 1, \dots, 4\}$ is given by [7]:

the two center distance:

$$\Delta z = z_2 - z_1 \quad (6)$$

the mass asymmetry:

$$\eta_A = \frac{A_1 - A_2}{A_1 + A_2} \quad (7)$$

the charge asymmetry:

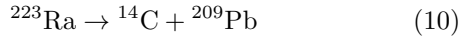
$$\eta_Z = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad (8)$$

the neck:

$$c_3 = \frac{1}{r_3} \quad (9)$$

where A_1, Z_1 and A_2, Z_2 are the number of nucleons and protons in the two daughter nuclei.

This choice of collective coordinates allows to describe a wide range of nuclear shapes involved in collective phenomena from a generalized point of view [17], e.g. a simultaneous description is made possible of general fission properties and the cluster-radioactive decay of radium



which was predicted by Sandulescu, Poenaru and Greiner in 1980 and later experimentally verified by Rose and Jones in 1984 [48, 50].

In order to describe the properties and dynamics of such a process, we start with the classical Hamiltonian function

$$H = T + V_0 \quad (11)$$

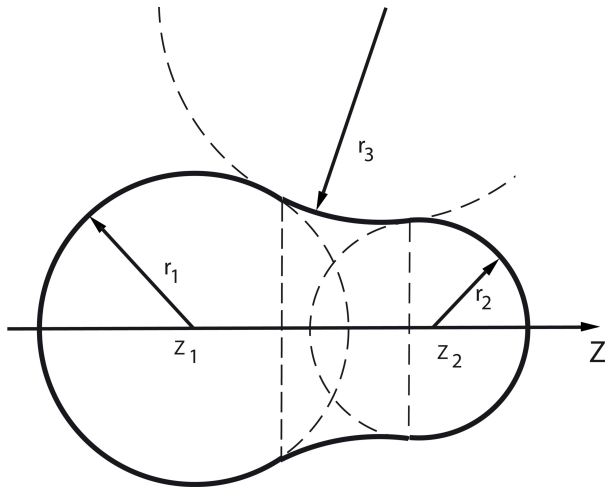


FIG. 1. The parametrization of the 3-spheres-model.

introducing a collective potential V_0 , depending on the collective coordinates,

$$V_0(q^i) = E_{\text{macro}}(q^i) + E_{\text{mic}}(q^i) \quad (12)$$

with a macroscopic contribution E_{macro} based on e.g. the liquid drop model and a microscopic contribution E_{mic} , which mainly contains the shell and pairing energy based on a single particle potential $V_{s.p.}$ and the classical kinetic energy T

$$T = \frac{1}{2} B_{ij}(q^i) \dot{q}^i \dot{q}^j \quad (13)$$

with collective mass parameters B_{ij} .

There are several common methods to generate the collective mass parameters B_{ij} , e.g. the cranking model [20] or irrotational flow models are used [23].

Quantization of the classical Hamiltonian[42] results in the collective Schrödinger equation

$$\hat{S}_0 \Psi(q^i, t) = \left(-\frac{\hbar^2}{2m_A} \frac{1}{\sqrt{B}} \partial_i B^{ij} \sqrt{B} \partial_j - i\hbar \partial_t + V_0 \right) \Psi(q^i, t) = 0 \quad (14)$$

with $B = \det B_{ij}$ is the determinant of the mass tensor. This is the central starting point for a discussion of nuclear collective phenomena.

For a specific realization of the single particle potential $V_{s.p.}$, for protons and neutrons respectively a Woods-Saxon type potential may be used. The advantages of such a potential are a finite potential depth and a given surface thickness. Furthermore arbitrary geometric shapes may be treated similarly by a folding procedure, which yields smooth potential values for such shapes.

For the three-spheres model, in order to define a corresponding potential, a Yukawa-function is folded with a given volume, which is uniquely determined within the model:

$$V_Y(\vec{r}) = -\frac{V_0}{4\pi a^3} \int_V d^3 r' \frac{\exp^{-|\vec{r}-\vec{r}'|/a}}{|\vec{r}-\vec{r}'|/a} \quad (15)$$

with the parameters potential depth V_0 and surface thickness a .

For protons, in addition the Coulomb-potential has to be considered, which is given for a constant density ρ_0

$$V_C(\vec{r}) = \frac{\rho_0}{a} \int_V d^3 r' \frac{1}{|\vec{r}-\vec{r}'|/a} \quad (16)$$

where the charge density is given by

$$\rho_0 = \frac{Ze}{\frac{4}{3}\pi R_0^3} \quad (17)$$

Both potentials may be written as general convolutions in R^3 of type:

$$V_{\text{type}}(\vec{r}) = C_{\text{type}} \int_{R^3} d^3 r' \rho(\vec{r}') w_{\text{type}}(|\vec{r}-\vec{r}'|) \quad (18)$$

with the weights

$$w_Y(d) = \frac{\exp^{-d/a}}{d/a} \quad (19)$$

$$w_C(d) = \frac{1}{d/a} \quad (20)$$

where $d = |\vec{x} - \vec{\xi}|$ is a measure of distance on R^3 and a density, which is constant inside the nucleus

$$\rho(\vec{r}) = \begin{cases} \rho_0 & \vec{r} \text{ inside the nucleus} \\ 0 & \vec{r} \text{ outside the nucleus} \end{cases} \quad (21)$$

Therefore the single particle potential $V_{s.p.}$ is given by

$$V_{s.p.} = V_Y + \left(\frac{1}{2} + t_3\right)V_C + \kappa \vec{\sigma}(\nabla V_Y \times \vec{p}) \quad (22)$$

with t_3 is the eigenvalue of the isospin operator with $+\frac{1}{2}$ for protons and $-\frac{1}{2}$ for neutrons, which guarantees that the Coulomb potential V_C only acts on protons. The last term is the spin-orbit term with the Pauli-matrices $\vec{\sigma}$, \vec{p} is the momentum operator and the strength is parametrized with κ . This term is necessary to split up the degeneracy of energy levels with different angular momentum and to generate the experimentally observed magic shell closures.

In the original Nilsson oscillator potential an additional \vec{l}^2 term was necessary to lower the higher angular momentum levels in agreement with experiment. For Woods-Saxon type potentials such a term is not necessary. Whether Riesz potentials are a realistic alternative, will be investigated in the next section.

The solutions of the single particle Schrödinger equation with the potential $V_{s.p.}$ yield the single particle energy levels, which are used to calculate the microscopic part of the total potential energy and contains two major parts, the shell and pairing corrections.

$$E_{\text{mic}}(q^i) = E_{\text{shell}}(q^i) + E_{\text{pair}}(q^i) \quad (23)$$

III. THE RIESZ POTENTIAL AS SMOOTH TRANSITION BETWEEN COULOMB AND FOLDED YUKAWA POTENTIAL

If we reinterpret the Riesz potential

$$V_{RZ}(\vec{r}) = C_{RZ} \int_{R^3} d^3r' \rho(\vec{r}') w_{RZ}(|\vec{r} - \vec{r}'|) \quad (24)$$

with the weight

$$w_{RZ}(d) = \frac{1}{(d/a)^\alpha} \quad 0 < \alpha < 3 \quad (25)$$

as the 3D-version of the general Riesz integral ([46]) applied to a scalar function $\rho(\vec{r})$, we may treat and interpret the Coulomb ($\alpha = 1$), Riesz- and Yukawa potentials similarly from a generalized point of view.

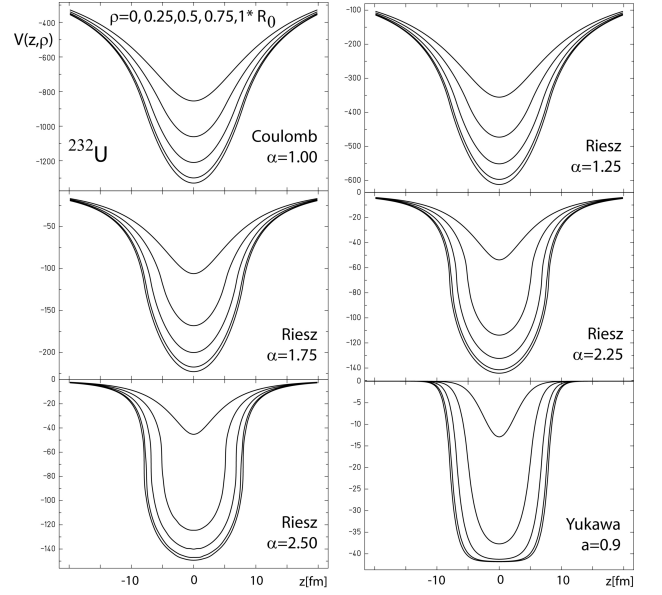


FIG. 2. For a spherical assumed shape (here ^{232}U) the potential for different weight functions is drawn. From top to bottom: Coulomb ($\alpha = 1.00$), Riesz ($\alpha = 1.25, 1.75, 2.25, 2.50$) and Yukawa (with $a = 0.9[\text{fm}]$) weight. In order to compare the plots with cylinder symmetric shapes, potential is drawn in cylinder coordinates (z, ρ) for a sequence of $\rho = 0.00, 0.25, 0.50, 0.75, 1.00 \times R_0$. $R_0(^{232}\text{U}) = 8.26[\text{fm}]$

In the following we will investigate the behavior of the Riesz potential with varying α , and compare its properties with the cases of Coulomb and Yukawa weight functions. In a way, the parameter α in the Riesz potential may be interpreted as a global screening of the Coulomb weight, such that the effect of the Yukawa exponential is partly modeled.

$$w_C(d) = \frac{1}{d/a} \quad (26)$$

$$w_{RZ}(d) = \frac{1}{(d/a)^{\alpha-1}} \frac{1}{d/a} \quad (27)$$

$$w_Y(d) = \exp^{-d/a} \frac{1}{d/a} \quad (28)$$

Therefore the Riesz potential could be an interesting alternative to the Yukawa potential in the case $\alpha \gg 1$. In a way, we expect the screening properties of the Riesz potential for increasing α to result in an interpolation between Coulomb and Yukawa limit.

Hence the fragmentation potentials used in a dynamic description of fission or cluster emission processes are an ideal framework to discuss and understand the properties of the Riesz integral.

The integral (18) with the weights (26)-(28) may be evaluated analytically for a spherical nucleus with radius R_0 .

$$\rho(r) = \rho_0 H(R_0 - r) \quad (29)$$

with the Heaviside step function H .

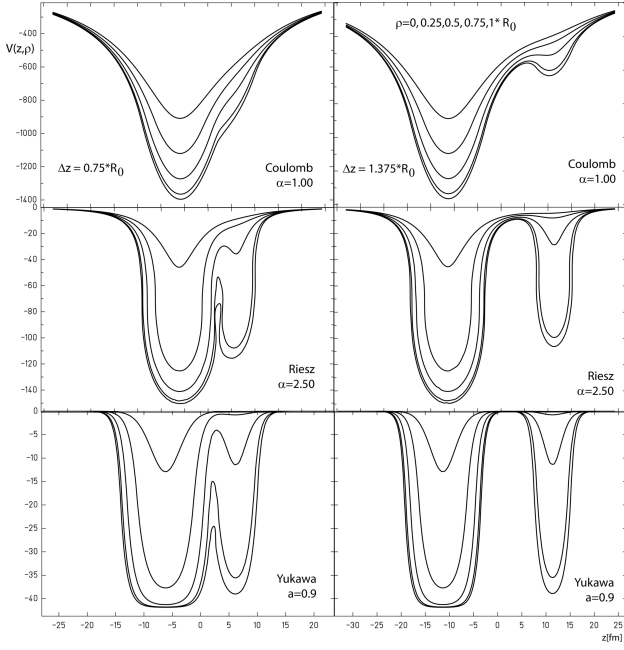


FIG. 3. For the configuration $^{232}\text{U} \rightarrow ^{208}\text{Pb} + ^{24}\text{Ne}$ the Coulomb, Riesz ($\alpha = 2.50$) and Yukawa ($a = 0.9[\text{fm}]$) is plotted for $\Delta z = 0.75R_0$ (left column) and $\Delta z = 1.375R_0$ (right column). $R_0(^{232}\text{U}) = 8.26[\text{fm}]$.

In this case we have:

$$V_{\text{type}}^{\text{sphere}}(r) = C_{\text{type}} \rho_0 \int_0^{R_0} r'^2 dr' \int_0^\pi \sin(\theta') d\theta' \times \int_0^{2\pi} d\phi' w_{\text{type}}(|\vec{r} - \vec{r}'|) \quad (30)$$

with

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta')} \quad (31)$$

With the substitution u

$$u = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta')} \quad (32)$$

we end up with a double integral for spherical shapes:

$$\begin{aligned} V_{\text{type}}^{\text{sphere}}(r) &= 2\pi C_{\text{type}} \rho_0 \int_0^{R_0} dr' r' / r \times \\ &\quad \int_{\sqrt{(r-r')^2}}^{\sqrt{(r+r')^2}} du u w_{\text{type}}(u) \quad (33) \\ &= 2\pi \rho_0 \frac{C_{\text{type}}}{r} \int_0^{R_0} dr' r' \times \\ &\quad \int_{|r-r'|}^{r+r'} du u w_{\text{type}}(u) \quad (34) \end{aligned}$$

This integral is valid for any analytic weight $w(u)$ and may be easily solved for the Coulomb, Riesz and Yukawa

weight functions. We obtain:

$$V_C^{\text{sphere}}(r) = aC_C \begin{cases} \frac{Ze}{R_0} \left(\frac{3}{2} - \frac{r^2}{2R_0^2} \right) & r \leq R_0 \\ \frac{Ze}{r} & r \geq R_0 \end{cases} \quad (35)$$

$$V_{\text{RZ}}^{\text{sphere}}(r) = 4\pi a^\alpha C_{\text{RZ}} \frac{2\pi}{(\alpha-2)(\alpha-3)(\alpha-4)} \frac{1}{r} \times \begin{cases} (r+R_0)^{3-\alpha}(r-(3-\alpha)R_0) + (R_0-r)^{3-\alpha}(r+(3-\alpha)R_0) & r \leq R_0 \\ (r+R_0)^{3-\alpha}(r-(3-\alpha)R_0) - (r-R_0)^{3-\alpha}(r+(3-\alpha)R_0) & r \geq R_0 \end{cases} \quad (36)$$

$$V_Y^{\text{sphere}}(r) = 4\pi a^3 C_Y \times \begin{cases} \left(1 - \left(1 + \frac{R_0}{a}\right)e^{-R_0/a} \frac{\sinh(r/a)}{r/a}\right) & r \leq R_0 \\ \frac{e^{-r/a}}{r/a} \left(\frac{R_0}{a} \cosh\left(\frac{R_0}{a}\right) - \sinh\left(\frac{R_0}{a}\right)\right) & r \geq R_0 \end{cases} \quad (37)$$

In figure 2 a sequence of these potentials is plotted for a spherical nucleus, ranging from Coulomb ($\alpha = 1.00$) and Riesz potential with increasing α up to the Yukawa potential with parameter settings according to [1].

For large $\alpha > 2.00$ the Riesz potential as well as the Yukawa potential model a finite surface thickness.

A remarkable difference between both potentials follows for small z . In this area, the Yukawa potential models a more Woods-Saxon type potential, while the Riesz potential may be compared with an harmonic oscillator potential. But this behavior is restricted only to the lowest energy levels; for realistic calculations the energy region near the Fermi-level is much more relevant. In this region, both potential types show a similar behavior for $2.00 < \alpha < 2.50$.

Hence both potentials seem interesting candidates for generation of realistic single particle energy levels.

For cylinder symmetric configurations the integral (18) cannot be solved analytically. Instead we switch to cylinder coordinates $\{\rho, z, \phi\}$. With the distance

$$d_{\text{cyl}} = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi') + (z - z')^2} \quad (38)$$

we have to solve the integral

$$V_{\text{type}}(\rho, z) = C_{\text{type}} \int_V d\rho' \rho' dz' d\phi' \rho(\rho', z') w_{\text{type}}(d_{\text{cyl}}) \quad (39)$$

numerically.

In figure 3 we have solved (39) and compare the three different weights for the strong asymmetric cluster decay

$$^{232}\text{U} \rightarrow ^{208}\text{Pb} + ^{24}\text{Ne} \quad (40)$$

The Riesz potential allows for a smooth transition between the Coulomb case and the Yukawa limit by varying α . Hence we obtain a direct geometric interpretation of the fractional parameter α .

Up to now, we discussed the properties of the single particle potential, which is the starting point for a calculation of the microscopic part of the collective potential.

On a macroscopic level the self energy of a given configuration contributes to the macroscopic part of the nuclear potential as the Coulomb- and surface or more sophisticated Yukawa energy term in a macroscopic energy formula, historically first used in Weizsäcker's famous liquid drop mass formula [60]:

$$E_{\text{macro}} = a_v A + a_s A^{2/3} - a_c Z A^{-1/3} + a_{\text{sym}}(N - Z)A^{-1} + a_{\text{pair}}A^{-1/2} \quad (41)$$

as a function of the nucleon number $A = r_0 R_0^3$ containing a volume, surface, Coulomb, symmetry and pairing term.

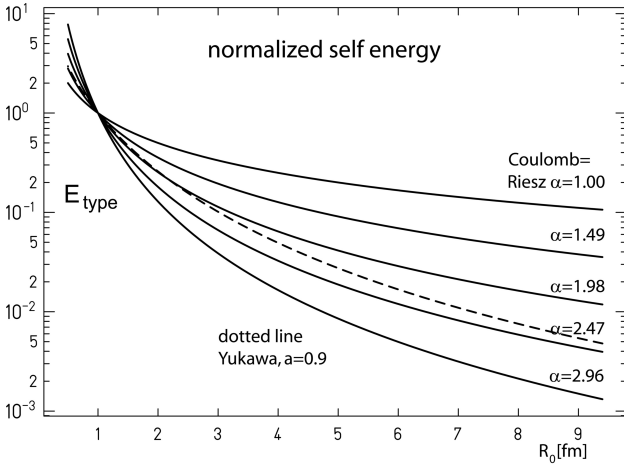


FIG. 4. For a spherical shape the self energy as a function of the sphere radius R_0 [fm] is plotted for the Coulomb, the Riesz ($\alpha = 1.49, 1.98, 2.47, 2.96$ and the Yukawa ($a = 0.9$ [fm]) is plotted. To compare all different types, all energies are normalized to $E_{\text{type}}(R_0 = 1) = 1$. Depending on R_0 , the Yukawa self energy lies is covered by the Riesz self energy within a range of $2.0 \leq \alpha \leq 2.5$ values

The self energy for a given charge type E_{type} is defined as the volume integral over the potential restricted on the volume of a given shape.

$$E_{\text{type}}(\vec{r}) = \frac{1}{2} C_{\text{type}} \int_V d^3 r \rho(\vec{r}) \int_{R^3} d^3 r' \rho(\vec{r}') w_{\text{type}}(|\vec{r} - \vec{r}'|) \quad (42)$$

or

$$E_{\text{type}}(\vec{r}) = \frac{1}{2} C_{\text{type}} \int_V \int_{V'} d^3 r d^3 r' \rho(\vec{r}) \rho(\vec{r}') w_{\text{type}}(|\vec{r} - \vec{r}'|) \quad (43)$$

For the 3 different weights (26)-(28) we obtain for the simplest case of a sphere with radius R_0 , a unit charge

($Ze = 1$) and with a thickness parameter $a > 0$:

$$E_C = \frac{3}{5} \frac{a}{R_0} \quad (44)$$

$$E_{RZ} = \frac{9 \times 2^{2-\alpha} a^\alpha}{(3-\alpha)(4-\alpha)(6-\alpha)} \frac{1}{R_0^\alpha}, \quad 0 \leq \alpha < 3 \quad (45)$$

$$E_Y = \frac{3a^3 - 3aR_0^2 + 2R_0^3 - 3ae^{-2R_0/a}(a + R_0)^2}{4R_0^6} \quad (46)$$

We obtain the important result that the Riesz self energy behaves like

$$E_{RZ} \sim \frac{1}{R_0^\alpha} \quad (47)$$

which scales with the nucleon number $A \sim R_0^3$ as

$$E_{RZ} \sim A^{-\alpha/3} \quad (48)$$

and therefore allows to model the influence of a screened Coulomb like charge contribution to the total energy.

In figure 4 we compare the R_0 dependence for the 3 different types of self energy. Depending on the size of the spherical nucleus e.g. $R_0(^{208}\text{Pb}) = 7.25$ the behavior of the Yukawa self energy is covered by the Riesz self energy for α within the range $2 \leq \alpha \leq 2.5$ and therefore the Riesz self energy covers the full range of relevant categories.

In addition it should be mentioned, that for the case $\alpha = 3/2$ the Riesz self energy behaves like $A^{-1/2}$ which emulates the pairing term and for $\alpha = 3$ the Riesz self energy behaves like A^{-1} which is equivalent to the behavior of the proton-neutron symmetry term in the Weizsäcker mass formula.

As a consequence, the pairing-, symmetry- and Coulomb contributions to the total energy content of a nucleus may be treated from a generalized view as different realizations of the same Riesz potential.

IV. CONCLUSION

From all these presented results we may draw the conclusion, that the Riesz potential may be considered as a promising alternative approach to folded potentials, which are widely used to describe nuclear dynamics within the framework of a collective shell model.

Of course, these potentials are only an alternative starting point to calculate fragmentation potentials based on a fractional integral definition, but it is indeed remarkable, that the Coulomb-, pairing- and symmetry part of the macroscopic energy contribution may be considered as specific realizations of the fractional Riesz integral with the fractional parameters $\alpha \in \{1, 3/2, 3\}$.

ACKNOWLEDGMENTS

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- [1] Bolsterli, M., Fiset, E. O., Nix, J. R. and Norton, J. L. (1972). *New calculation of fission barriers for heavy and superheavy nuclei* Phys. Rev. C **5**, 1050–1077, doi:10.1103/PhysRevC.5.1050
- [2] Brack, M. (1993). *The physics of simple metal clusters: self-consistent jellium model and semi-classical approaches* Rev. Mod. Phys. **65**, 677–732, doi:10.1103/RevModPhys.65.677
- [3] Chowdhury, P. R., Samanta, C. and Basu, D. N. (2006). *α decay half-lives of new super-heavy elements* arXiv:nucl-th/0507054v2, Phys. Rev. C **73**, 014612, doi:10.1103/PhysRevC.73.014612
- [4] Clemenger, K. (1985). *Ellipsoidal shell structure in free-electron metal clusters* arXiv:1207.7235 [hep-ex], Phys. Rev. B **32**, 1359–1362, doi:10.1103/PhysRevB.32.1359
- [5] Davies, K. T. R., Sierk, A. J. and Nix J. R. (1976). *Effect of viscosity on the dynamics of fission* Phys. Rev. C **13**(6), 2385–2403, doi:10.1103/PhysRevC.13.2385
- [6] Depta, K., Herrmann, R., Greiner, W., Scheid, W. and Sandulescu, A. (1985). *Fission mass yields of excited-medium heavy nuclei studied within the fragmentation theory - the ^{172}Yb as an example* J. Phys. G: Nucl. Part. Phys. **11**, 1087, doi:10.1088/0305-4616/11/9/015
- [7] Depta, K., Greiner, W., Maruhn, J. A., Wang, H. J., Sandulescu, A. and Herrmann, R. (1990). *On the necking-in process in cluster decays* Intern. Journal of Modern Physics A, **5**(20), 3901–3928, doi:10.1142/S0217751X90001677
- [8] Eisenberg, J. M. and Greiner, W. (1987). *Nuclear models* North Holland, Amsterdam
- [9] Elsassner, W. M. (1933). *Sur le principe de Pauli dans les noyaux* J. Phys. Radium **4**, 549–556, *ibid.* **5** (1934), 389–397, *ibid.* **5** (1934), 635–639, doi:10.1051/jphysrad:01934005012063500, <http://hal.archives-ouvertes.fr/jpa-00233288>
- [10] Engel, E., Schmitt, U. R., Lüdde, H.-J., Toepfer, A., Wüst, E., Dreizler, R. M., Knospe, O., Schmidt, R. and Chattopadhyay, P. (1993). *Accurate numerical study of the stability of Na_{19} -cluster dimers* Phys. Rev. B **48**, 1862–1869, doi:10.1103/PhysRevB.48.1862
- [11] Fink, H. J., Maruhn, J., Scheid, W. and Greiner, W. (1974). *Theory of fragmentation dynamics in nucleus-nucleus collisions* Z. Physik **268**, 321–331, doi:10.1007/BF01669469
- [12] Gneuss, G. and Greiner, W. (1971). *Collective potential energy surfaces and nuclear structure* Nucl. Phys. A **171**, 449–479, doi:10.1016/0375-9474(71)90596-3
- [13] Greiner, W. and Maruhn, J. A. (1997). *Nuclear models* Springer, Berlin, Heidelberg, New York
- [14] Greiner, W., Park, J. A. and Scheid, W. (1995). *Nuclear molecules* World Scientific Publ., Singapore
- [15] Grumann, J., Mosel, U., Fink, B. and Greiner, W. (1969). *Investigation of the stability of superheavy nuclei around $Z=114$ and $Z=164$* Z. Phys. **228** 371–386, doi:10.1007/BF01406719
- [16] Heer, W. A. de (1993). *The physics of simple metal clusters: experimental aspects and simple models* Rev. Mod. Phys. **65**(3), 611–676, doi:10.1103/RevModPhys.65.611
- [17] Herrmann, R., Maruhn, J. A. and Greiner, W. (1985). *Towards a unified description of asymmetric nuclear shapes in structure, fission and cluster radioactivity* J. Phys. G: Nuclear Physics **12**(12), L285–L290, doi:10.1088/0305-4616/12/12/004
- [18] Herrmann, R., Depta, K., Schnabel, D., Klein, H., Renner, W., Poenaru, D. N., Sandulescu, A., Maruhn, J. A. and Greiner, W. (1988). *Nuclear deformation, cluster-structure, fission and cluster radioactivity* in Märtens, H. and Seeliger, D. (Eds.) (1988). *Physics and chemistry of fission, Proceedings of the XVIIIth international symposium on nuclear physics devoted to the fiftieth anniversary of the discovery of nuclear fission* Zentralinstitut für Kernforschung, Rossendorf/Dresden, ZfK-732, indc-special 6, pp. 191–211, available online at IAEA, Vienna, Austria, nuclear data services, indc-special series - indc reports <http://www-nds.iaea.org/reports-new/indc-reports/indc.../indcspecial.pdf>
- [19] Herrmann, R. (2011). *Fractional calculus- An introduction for physicists* World Scientific, Singapore
- [20] Inglis, D. R. (1954). *Particle derivation of nuclear rotation properties associated with a surface wave* Phys. Rev. **96**, 1059–1064, doi:10.1103/PhysRev.96.1059
- [21] Iwamoto, A. and Herrmann, R. (1991). *Evaporation of charged particles from highly deformed nucleus* Z. Phys. A: Hadrons and Nuclei **338**(3), 303–307, doi:10.1007/BF01288194
- [22] Jackson, J. D. (1998). *Classical electrodynamics* 3rd ed. Wiley, New York
- [23] Kelson, I. (1964). *Dynamic calculation of fission of an axial symmetric liquid drop* Phys. Rev. **136**, B1667–B1673, doi:10.1103/PhysRev.136.B1667
- [24] Kibble, T. W. B. and Berkshire, F. H. (2004). *Classical mechanics* Imperial College Press, London
- [25] Knight, W. D., Clemenger, K., de Heer, W. A., Saunders, W. A., Chou, M. Y. and Cohen, M. L. (1984). *Electronic shell structure and abundances of sodium clusters* Phys. Rev. Lett. **52**, 2141–2143, doi:10.1103/PhysRevLett.52.2141
- [26] Krappe, H. J. and Pomorski, K. (2012). *Theory of nuclear fission* Lecture notes on physics **838** Springer, Berlin, Heidelberg, New York, doi:10.1007/978-3-642-23515-3
- [27] Kruppa, A. T., Bender, M., Nazarewicz, W., Reinhard, P. G., Vertse, T. and Cwiok, S. (2000). *Shell corrections of superheavy nuclei in self-consistent calculations* Phys. Rev. C **61**, 034313–034325, doi:10.1103/PhysRevC.61.034313
- [28] Lustig, H.-J., Maruhn, J. A. and Greiner, W. (1980). *Transitions in the fission mass distributions of the fermium isotopes* J. Phys. G **6**, L25–L36, doi:10.1088/0305-4616/6/2/001
- [29] Martin, T. P., Bergmann, T., Göhlich, H. and Lange, T. (1991). *Electronic shells and shells of atoms in metallic clusters* Z. Phys. D **19**, 25–29, doi:10.1007/BF01448248
- [30] Maruhn, J. A. and Greiner, W. (1972). *The asymmetric two center shell model* Z. Physik **251**, 431–457, doi:10.1007/BF01391737
- [31] Maruhn, J. A., Hahn, J., Lustig, H.-J., Ziegenhain, K.-H. and Greiner, W. (1980). *Quantum fluctuations within the fragmentation theory* Prog. Part. Nucl. Phys. **4**, 257–271, doi:10.1016/0146-6410(80)90009-5
- [32] Miller, K. and Ross, B. (1993). *An introduction to fractional calculus and fractional differential equations* Wiley, New York.
- [33] Möller P. and Nix, J. R. (1981). *Nuclear mass formula*

- with a Yukawa-plus-exponential macroscopic model and a folded-Yukawa single-particle potential Nucl. Phys. A **361**(1) 117–146, doi:10.1016/0375-9474(81)90473-5
- [34] Möller P., Nix, J. R., Myers, W. D. and Swiatecki, W. J. (1993). *Nuclear ground-state masses and deformations* arXiv:nucl-th/9308022v1, Atomic Data Nucl. Data Tables **59**(2), (1995) 185–381, doi:10.1006/adnd.1995.1002
- [35] Möller, P., Bengtsson, R., Carlsson, B. G. Olivius, P., Ichikawa, T. Sagawa, H. and Iwamoto, A. (2008). *Axial and reflection asymmetry of the nuclear ground state* Atomic Data and Nuclear Data Tables **94**(5), 758–780, doi:10.1016/j.adt.2008.05.002
- [36] Mosel, U. and Greiner, W. (1969). *On the stability of superheavy nuclei against fission* Z. Phys. A **222**, 261–282, doi:10.1007/BF01392125
- [37] Myers, W. D. and Swiatecki, W. J. (1966). *Nuclear masses and deformations* Nucl. Phys. **81**, 1–60, doi:10.1016/0029-5582(66)90639-0
- [38] Nilsson, S. G. (1955). Kgl. Danske Videnskab. Selsk. Mat.-Fys. Medd. **29**, 431.
- [39] Nilsson, S. G., Tsang, C. F., Sobiczewski, A., Szyma, Z., Wycech, S., Gustafson, C., Lamm, I., Möller, P. and Nilsson, B. (1969). *On the nuclear structure and stability of heavy and super-heavy elements* Nucl. Phys. A **131**, 1–66, doi:10.1016/0375-9474(69)90809-4
- [40] Ortigueira, M. D. (2006). *Riesz potential operators and inverses via fractional centred derivatives* International Journal of Mathematics and Mathematical Sciences, Article ID 48391, 1–12, doi:10.1155/IJMMS/2006/48391
- [41] Podlubny, I. (1999). *Fractional differential equations* Academic Press, New York
- [42] Podolsky, B. (1928). *Quantum-mechanically correct form of Hamiltonian function for conservative systems* Phys. Rev. , **32**(5), 812–816. doi:10.1103/PhysRev.32.812
- [43] Poenaru, D. N., Gherghescu, R. A. and Greiner, W. (2010a). *Individual and collective properties of fermions in nuclear and atomic cluster systems* J. Phys. G: Nucl. Part. Phys. **37**, 085101, doi:10.1088/0954-3899/37/8/085101
- [44] Poenaru, D. N. and Greiner, W. (2010b). *Cluster radio activity* in Beck, C. (Ed.) *Clusters in Nuclei I* 1–56, Springer, Berlin, Heidelberg, New York, doi:10.1007/978-3-642-13899-7_1
- [45] Reinhard, P. G. and Suraud, E. (2004). *Introduction to cluster dynamics* Wiley-VCH, Weinheim, Germany
- [46] Riesz, M. (1949). *L'intégrale de Riemann-Liouville et le problème de Cauchy* Acta Math. **81**, 1–222
- [47] Ring, P. and Schuck, P. (2008). *The nuclear many-body problem* Springer Berlin, Heidelberg, New York
- [48] Rose, H. J. and Jones, G. A. (1984). *A new kind of natural radioactivity* Nature **307**, 245–247, doi:10.1038/307245a0
- [49] Sandulescu, A., Gupta, R. K., Scheid, W. and Greiner W. (1976). *Synthesis of new elements within the fragmentation theory: Application to $Z = 104$ and 106 elements* Phys. Lett. B **60**(3), 225–228, doi:10.1016/0370-2693(76)90286-0
- [50] Sandulescu, A., Poenaru, D. N. and Greiner W. (1980). *New type of decay of heavy nuclei intermediate between fission and alpha decay* Soviet Journal of Particles and Nuclei **11** 528–541
- [51] Scharnweber, D., Mosel, U. and Greiner, W. (1970). *Asymptotically correct shell model for nuclear fission* Phys. Rev. Lett. **24**, 601–603, doi:10.1103/PhysRevLett.24.601
- [52] Sobiczewski, A. and Pomorski, K. (2007). *Description of structure and properties of super-heavy nuclei* Prog. Part. Nucl. Phys. **58**, 292–349, doi:10.1016/j.pnpnp.2006.05.001
- [53] Staszczak, A., Baran, A. and Nazarewicz, W. (2012). *Spontaneous fission modes and lifetimes of super-heavy elements in the nuclear density functional theory* arXiv:1208.1215 [nucl-th], Phys. Rev. C **87**, 024320 (2013) [7 pages], doi:10.1103/PhysRevC.87.024320
- [54] Strutinsky, V. M. (1967a). *Microscopic calculation of the nucleon shell effects in the deformation energy of nuclei* Ark. Fysik **36**, 629–632
- [55] Strutinsky, V. M. (1967b). *Shell effects in nuclear masses and deformation energies* Nucl. Phys. A **95**(2), 420–442, doi:10.1016/0375-9474(67)90510-6
- [56] Strutinsky, V. M. (1968). *Shells in deformed nuclei* Nucl. Phys. A **122**, 1–33, doi:10.1016/0375-9474(68)90699-4
- [57] Tajima, N. and Suzuki, N. (2001). *Prolate dominance of nuclear shape caused by a strong interference between the effects of spin-orbit and l^2 terms of the Nilsson potential* Phys. Rev. C **64**, 037301, doi:10.1103/PhysRevC.64.037301
- [58] Troltenier, D., Maruhn, J. A., Greiner, W., Velazquez Aguilar, V., Hess, P. O. and Hamilton, J. H. (1991). *Shape transitions and shape coexistence in the Ru and Hg chains* Z. Phys. A **338**(3), 261–270, doi:10.1007/BF01288188
- [59] Vasak, D., Shanker, R., Müller, B. and Greiner, W. (1983). *Deformed solutions of the MIT quark bag model* J. Phys. G: Nucl. Phys. **9**, 511–520, doi:10.1088/0305-4616/9/5/004
- [60] Weizsäcker, C. F. v. (1935). *Zur Theorie der Kernmassen* Z. Phys. **96**, 431–458
- [61] Werner, F. G. and Wheeler, J. A. (1958). *Superheavy nuclei* Phys. Rev. **109**, 126–144, doi:10.1103/PhysRev.109.126
- [62] Wu, X. Z., Depta, K., Herrmann, R., Maruhn, J. A. and Greiner, W. (1985). *Study of dynamics in ternary fission by solving classical equations of motion II* Nuovo Cimento, **87**(3), 309–323, doi:10.1007/BF02902224